17. Show that every matrix A in $\mathbb{R}^{n\times m}$ admits a singular value decomposition

$$A = U\Sigma V^*$$

with orthogonal (unitary) matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$, and with $\Sigma \in \mathbb{R}^{n\times m}$ a generalized diagonal matrix ($\Sigma_{i,j} = 0\ \forall i \neq j$, i.e. the square part is diagonal and the rest of the rectangle is zero), where:

- the matrix $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_m)$ of singular values is unique if we require $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m$, and
- if A is square and if the singular values are pairwise distinct, i.e. $\sigma_1 > \sigma_2 > \ldots > \sigma_m$, the singular vectors u_i and v_j (the columns of U resp. V) are unique up to the sign (i.e. up to scalar factors λ with $|\lambda| = 1$)

(and likewise for $\mathbb{C}$ instead of $\mathbb{R}$, which should follow from the same proof steps).

18. Let $A \in \mathbb{R}^{n\times m}$ or $\mathbb{C}^{n\times m}$ with $n > m$ be a rectangular matrix of full rank $\operatorname{rk}(A) = m$ and $b \in \mathbb{R}^n$ resp. $\mathbb{C}^n$. The **least squares solution** x of the overdetermined system of linear equations $Ax = b$ is the vector $x \in \mathbb{R}^m$ resp. $\mathbb{C}^m$ which minimizes $\|Ax - b\|_2^2$. Show that the least squares solution is the solution x of $A^*Ax = A^*b$. How can we use a singular value decomposition of A to compute it?

19. Let $A \in \mathbb{R}^{n\times m}$ or $\mathbb{C}^{n\times m}$ with $n > m$ be a rectangular matrix of full rank $\operatorname{rk}(A) = m$, and let $P = (A^*A)^{-1}A^*$ (an $m \times n$ matrix).

    It is obvious that $PA = \mathbb{I}$, the $m \times m$ identity matrix. Because of this identity, P is called the *pseudoinverse* of A, denoted $A^\dagger$.

    (a) What is the relation between the pseudoinverse and the least-squares problem from example 18?

    (b) Describe how $P = A^\dagger$ can be computed efficiently using a singular value decomposition of A.

20. (a) Let $A \in \mathbb{R}^{n\times m}$ or $\mathbb{C}^{n\times m}$ be a matrix of rank $\operatorname{rk}(A) = r < m$ and $b \in \mathbb{R}^n$ resp. $\mathbb{C}^n$. The least squares solution x of the system of linear equations $Ax = b$ is the vector $x \in \mathbb{R}^m$ resp. $\mathbb{C}^m$ with minimal norm $\|x\|_2$ which minimizes $\|Ax - b\|_2^2$. Let $A = U\Sigma V^*$ be a singular value decomposition of A, u_i the columns of U, σ_i the singular values and v_i the columns of V. Show that the least squares solution is given by

$$x = \sum_{i=1}^{r} \frac{\langle u_i, b\rangle}{\sigma_i} v_i.$$

    (b) Let $A \in \mathbb{R}^{n\times m}$ or $\mathbb{C}^{n\times m}$ with $n < m$ be a rectangular matrix of full rank $\operatorname{rk}(A) = n$ and $b \in \mathbb{R}^n$ resp. $\mathbb{C}^n$. Deduce from part (a) that the solution x of the underdetermined system of linear equations $Ax = b$ with minimal norm $\|x\|_2$ is given by

$$x = \sum_{i=1}^{n} \frac{\langle u_i, b\rangle}{\sigma_i} v_i$$

    where u_i, σ_i and v_i are defined as in part (a).

21. Compute (by hand) the singular value decomposition of the matrix

$$\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 3 & 0 & 8 & 0 \end{pmatrix}.$$

22. Write a program in MATLAB or Octave which takes as input parameter a (not necessarily square) matrix and transforms it into bidiagonal form using the **Golub-Kahan bidiagonalization**.

23. Write a program in MATLAB or Octave which takes as input parameter a (not necessarily square) matrix and transforms it into bidiagonal form using the **LHC bidiagonalization**.

24. (a) Write a program in MATLAB or Octave which takes as input parameter a (not necessarily square) matrix and transforms it into bidiagonal form using the **three step bidiagonalization**. (You should be able to reuse your code from examples 22 and 23.)

    (b) Compare the execution times of the 3 different methods on differently-shaped matrices. Do your experimental speed results match the theory?

25. Write a program in MATLAB or Octave which takes as input parameter a (not necessarily square) bidiagonal matrix and computes its singular values using the **implicit QR method** (with shifts and both kinds of deflation steps). Test the program on differently-shaped bidiagonal matrices (e.g. the outputs of your tests in example 24 (b)).